# Quick Guide to $\mathbb{C}$omplex Numbers

For Ryan Holben's Math 3D class, Winter 2015 at UC Irvine, last updated January 15th, 2015.

## 0.1 Basic definition

First we define the imaginary unit

$$i = \sqrt{-1}.$$

The number $i$ has several useful properties that you can derive, including:

$$i^2 = -1 \text{ and } \frac{1}{i} = -i.$$

**Imaginary numbers** are numbers of the form $bi$, where $b \in \mathbb{R}$. Next, we can define **complex numbers**, which are numbers of the form

$$z = a + bi,$$

where $a, b \in \mathbb{R}$. The shorthand way of saying "$z$ is a complex number" is to simply write $z \in \mathbb{C}$. We can break a complex number $z = a + bi$ down into its components. The **real part** of $z$ is $\text{Re}(z) = \text{Re}(a+bi) = a$, and the **imaginary part** of $z$ is $\text{Im}(z) = \text{Im}(a + bi) = b$.

Note that another way of representing complex numbers is with vectors $(a, b)$ in the complex plane, where the horizontal axis corresponds to real numbers, and the vertical axis corresponds to imaginary numbers.

### 0.1.1 Example: Finding complex roots

At the most basic level, complex numbers are useful because they allow us to find roots for all polynomials. For example, if

$$z^2 + 4 = 0$$

then we can solve for two roots:

$$z^2 = -4$$
$$z = \pm\sqrt{-4} = \pm\sqrt{-1 \cdot 4} = \pm\sqrt{4}\sqrt{-1} = \pm 2i$$

### 0.1.2 Example: Fractions

We usually prefer to write complex numbers in the form $a + bi$. What if we have a fraction? For example:

$$\frac{1}{2+3i}$$

Then we rationalize it:

$$\frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{2-3i}{(2+3i)(2-3i)}$$
$$= \frac{2-3i}{4-9i^2} = \frac{2-3i}{4+9}$$
$$= \frac{2-3i}{13} = \frac{2}{13} - \frac{3}{13}i$$

## 0.2 Euler's formula

Euler's formula relates complex exponentials to sine and cosine:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This identity can be proved using Taylor series, and it is very important. As a side note, from this formula you can see that $e^{i\pi} = -1$, a widely popularized formula which you may have seen before.

Euler's formula lets us define sine and cosine in terms of complex exponentials:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$